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THE USE OF DISTRIBUTION FUNCTIONS TO REPRESENT UTILITY FUNCTIONS

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FACULTY WORKING PAPERS

College of Commerce and Business Administration

University of Illinois at Urbana - Champaign

March 8, 1971

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No. 7

The Use of Distribution Functions to Represent Utility Functions

We are considering an individual whose evaluation and consequent choice of actions is accomplished through the use of the expected utility hypothesis [7]. For our purposes it is assumed, that the individual has uncertainty only with respect to a dollar amount w. Letting v(w) be the individual's utility on the dollar amount w, we will make the common assumption that this utility function is increasing on w with upper and lower bounds. It is then clear that there exists a linear transformation on any utility function of this form which will give us the utility function in the form of a distribution function. Since linear transformations do not affect either the individual's choices or his certainty equivalent then any utility function satisfying the above restrictions could be converted to a distribution function without altering the individual's preferences for various decisions.

As it turns out there are advantages to having the utility function represented by a distribution function. In the first place there are a wide variety of standard distribution functions so sufficient flexibility exists to adapt this concept to a significant number of applications. In addition, these utility functions are easier to communicate, can serve as first approximations, form a basis for illustration, and are useful in easing the analysis of many types of problems. The widely used exponential utility function, [5, p. 255] and [2, p. 90] among many others, can be viewed as a distribution function of the exponential density function.

This paper exploits the use of standard distribution functions by taking advantage of their mathematical properties to determine the individual's certainty equivalent (fixed amount for which he would be indifferent between it and the uncertain payoff) as a function of the parameters of the distribution function and the parameters of the density function on the uncertain end state w. The parameters of the density function on w are, to some extent, controlled by the individual through his choice of a decision. Thus the individual will choose an action that is associated with the parameters resulting in an optimal certainty equivalent. For this reason it is desirable to have the certainty equivalent as a function of these parameters.

In addition we relate our results to the risk premium (expected monetary value minus certainty equivalent). The risk premium is a widely used concept, [2], [8] and [10] among others. It is useful as a measure of risk aversion although it is not the only measure of risk aversion, see [1], [4] and [8].

The results are also useful for purposes of estimation. If we assume that an individual's utility function is from a specific family of distribution functions and in addition we have data on certainty equivalents for various gambles along with the probability distributions on the payoffs then we can estimate the parameters of that individual's utility function. This procedure was used by Berhold [2] to estimate the utility function for a large government contractor under the assumption that it was an exponential utility function.

As we said, the individual is, in effect, choosing the parameters of the probability distribution on w. For a specific class or type of probability distributions (Gaussian, binomial, etc.) we can characterize the probability distribution by certain parameters (σ and μ for the Gaussian distribution). We are letting a vector, denoted by a, index these distributions, thus $a = [a_1, a_2 \ldots, a_n]$. For example, with the Gaussian distribution, we could let $a_1 = \mu$ and $a_2 = \sigma$, where $n = 2$. In general we will let $w = x(a;z)$ where the probability distribution on z is not a function of the decision. For example, z can have a standard Gaussian distribution (zero mean and unit variance) and let $x(a;z) = z\sigma + \mu$.

Next we will augment the individual's payoff by a money amount h(a) (positive or negative). So his net payoff will be

$$1) \quad x(a;z) + h(a)$$

where $h(a^\circ) = 0$ and a° is an arbitrary vector. Later we will make a° more specific. We define h(a) as a dollar amount such that

$$2) \quad Ev\left[x(a;z) + h(a)\right] = Ev\left[x(a^\circ;z) + h(a^\circ)\right].$$

So in taking the partial derivative of (2) we get

$$3) \quad E\left\{v'\left[x(a;z) + h(a)\right]\left[\frac{\partial x(a;z)}{\partial a_i} + \frac{\partial h(a)}{\partial a_1}\right]\right\} = 0$$

for all i, [i=1, ---, n], where the $p(z|a)$ is a constant for all a. This will not restrict us to any extent. And since $\partial h(a)/\partial a_1$ is constant for

all z, then

$$4) \quad \frac{\partial h(a)}{\partial a_i} = - \frac{E\{\partial x(a;z)/\partial a_i \; v'[x(a;z) + h(a)]\}}{E\{v'[x(a;z) + h(a)]\}}$$

We can now make an observation that allows us to derive some significant results in addition to providing a basis for subsequent developments along the same line. Using the relationship in (4) we want to exploit the general result that

$$5) \quad \frac{v'(K)\; p(K)}{E[v'(K)]}$$

satisfies the requirements of a probability measure on K (non-negative, sums to 1), since v (·) is strictly increasing in money and has upper and lower bounds. This leads us to the consideration of distribution functions as utility functions.

Using well-known distribution functions would result in v'(·) being a well-known density function. Thus, we could use the known characteristics of the density function as well as its tabled values. The distribution function has the desirable characteristics of being strictly increasing and having upper and lower bounds. These characteristics are commonly required of utility functions defined on money amounts.

So assuming the utility function is a distribution function, we can then view v'[x(a;z) + h(a)] as a conditional probability distribution on a given z, say p(a|z). This is not to say that there is actually a conditional probability distribution on a. We merely want to take

advantage of the fact that it can satisfy the requirements of a probability measure.

For the time being let us adopt the fiction that $v'(\cdot;z)$ is a probability distribution on the action a given z. Then in terms of Bayes Theorem we get

$$6) \quad \pi(z|a) = \frac{v'[x(a'z) + h(a)]\, p(z)}{E\, v'\,[x(a;z) + h(a)]}$$

where $\pi(z|a)$ is tne posterior density function. Thus we can take advantage of results for posterior density functions, see [9] and [10].

As an illustration let the Gaussian distribution function represent the utility function with parameters μ_o and σ_o. Let the profit have a Gaussian distribution with parameters μ and σ and $a = [a_1, a_2]$ where $a_1 = \mu$ and $a_2 = \sigma$. Thus, we have $p(z)$ as the standard Gaussian distribution with a standard deviation of 1 and a mean of zero. The profit function is

$$7) \quad x(\mu, \sigma;z) = \mu + z\sigma.$$

We will let a° be $a_1^\circ = \mu^\circ$ and $a_2^\circ = \sigma^\circ$.

When $x(a;z)$ has a normal distribution it is immediately clear that

$$8) \quad h(\mu,\sigma) = g(\sigma;\mu) - \mu + \mu_1^\circ$$

where μ is used to index the function $g(\sigma)$. So, using (6) we next get $d\, g(\sigma;\mu) = - E[z\pi(z|a)\, d\sigma]$.

Next we will determine $\pi(z|a)$. The constants will be ignored until the end because we know $\pi(z|a)$ is a density function and it will turn out to be a well-known type. Since the denominator of (6) is a constant, it can be immediately dropped. Thus,

$$9) \quad \pi(z|\sigma) \propto \exp\left\{-1/2\left[(z-M)/S\right]^2\right\} \exp\left\{-1/2\, z^2\right\},$$

where $M = [\mu_o - \mu - g(\sigma;\mu)]/\sigma$

$S = \sigma_o/\sigma.$

To obtain the mean of the posterior distribution we use a result from Schlaifer [11, p. 441] and get

$$10) \quad d\,g(\sigma;\mu) = -E\left\{z\,\pi(z\,a)\right\} = -\frac{M/S^2}{1 + (1/S^2)} = -M/(S^2+1).$$

$$= -\frac{\sigma[\mu_o - \mu - g(\sigma;\mu)].}{\sigma_o^2 + \sigma^2}$$

Rearranging we get

$$11) \quad \frac{d\,g(\sigma;\mu)}{g(\sigma;\mu) + \mu - \mu_o} = \frac{\sigma\, d\,\sigma}{\sigma_o^2 + \sigma^2}$$

and on integrating

$$12) \quad \text{Log}\,[g(\sigma;\mu) + \mu - \mu_o] \Big|_o^{\sigma_*} = 1/2\,\log\,[\sigma_o^2 + \sigma^2] \Big|_o^{\sigma_*}$$

And since $g(0;\mu) = 0$ then

$$13) \quad \text{Log}\,\frac{g(\sigma_*;\mu) + \mu - \mu_o}{\mu - \mu_o} = \text{Log}\left[1 + \left(\frac{\sigma_*}{\sigma_o}\right)^2\right]^{1/2}$$

In the following we will deal only with σ_* so to simplify we will drop the *. Consequently,

$$14) \quad g(\sigma;\mu) = \{\mu - \mu_o\}\left\{\left[1 + \left(\frac{\sigma}{\sigma_o}\right)^2\right]^{1/2} - 1\right\}.$$

We would like to pause here a moment in order to relate $g(\sigma;\mu)$ to the concept of a risk premium. The risk premium is the expected monetary value minus the certainty cash equivalent. The certainty cash equivalent or simply certainty equivalent is a fixed money amount c(a) such that the individual is indifferent between choosing either c(a) or the uncertain amount [x(a;z) + h(a)]. In other words c(a) is a dollar amount such that

$$15) \quad Ev\,[x(a;z) + h(a)] = v\,[c(a)].$$

It is clear that $c(a) = \mu$ when $a = [\mu, 0]$. The risk premium is a money amount R(a) where

$$16) \quad R(a) \equiv E\,[x(a;z) + h(a)] - c(a)$$

So from (16) and (8) we get

$$17) \quad c(\mu,\sigma) = \mu + g(\sigma;\mu) - \mu + \mu^o - R(\mu,\sigma) = c(\mu^o,0)$$

$$= \mu^o$$

Consequently,

$$18) \quad R(\mu,\sigma) = g(\sigma;\mu)$$

which holds for all cases where x(a,z) has a normal distribution.

The result in (14) confirms our intuition in the following manner: We would intuitively expect $R(\sigma,\mu)$ to be strictly increasing (decreasing) in σ_o if $\mu > (<) \mu_o$. We would also expect $R(\sigma,\mu)$ to be strictly decreasing (increasing) in σ_o if $\mu > (<) \mu_o$. In fact, $\text{Lim}_{\sigma_o \to o} R(\sigma,\mu) = \mu - \mu_o$ and $\text{Lim}_{\sigma_o \to \infty} R(\sigma,\mu) = 0$. And if $\sigma_o \approx 0$, then $P[v < \varepsilon] + P[v < 1-\varepsilon] \approx 1$, $\varepsilon > 0$. In other words, there are two outcomes (in an approximate sense); maximum utility ($v = 1$) and minimum utility ($v = 0$). Taking the partial derivative of (14) with respect to μ_o we get

$$19) \quad \frac{\partial\, g(\sigma;\mu)}{\partial \mu_o} = \left\{1 - \left[1 + \left(\frac{\sigma}{\sigma_o}\right)^2\right]^{1/2}\right\} < 0.$$

Verbally the individual has a lower risk premium as μ_o increases. Taking the partial derivative with respect to σ_o we get

$$20) \quad \frac{\partial\, g(\sigma;\mu)}{\partial \mu_o} = -[\mu - \mu_o]\left[1 + \left(\frac{\sigma}{\sigma_o}\right)^2\right]^{1/2}\left[\frac{\sigma^2}{\sigma_o}\right] < 0.$$

Verbally the individual has a lower risk premium as σ_o increases.

Besides being strictly increasing, having upper and lower bounds, the Gaussian distribution also has two other characteristics: I. Two parameters, μ_o and σ_o, which can be used to define an individual's utility function and II. A risk aversive portion ($\mu > \mu_o$) and a risk preference portion ($\mu < \mu_o$). The Gaussian utility function (as denoted here) has sufficient flexibility in terms of the above characteristics to match or sufficiently approximate utility functions for many individuals. In addition, it is a well-known function and widely tabulated. Thus, the Gaussian utility function could be of immediate use.

To further illustrate: if

$$21) \quad v(k) = 1 - 1/2(\eta)\exp[-2\eta K], \text{ where } \eta > 0,$$

then the agent has an exponential utility function. Observe that if $\eta = 1/2$, then this is a distribution function on $K \geq 0$. Proceeding as before we get

$$22) \quad \pi(z|a) \propto \exp\{-2\eta[z\sigma + \mu + g(\sigma;\mu)]\} \exp\{-z^2/2\}$$

$$= \exp\{-M/2\}$$

where $M = z^2 + 4\eta z\sigma + 4\eta M + 4\eta g(\sigma;\mu)$

Rearranging and completing the square we get

$$23) \quad M = z^2 + 4\eta z\sigma + (2\eta\sigma)^2 - (2\eta\sigma)^2 + 4\eta M + 4\eta g(\sigma;\mu).$$

The last three terms are constants for a given value of σ so they drop out. Thus,

$$24) \quad \pi(z|a) \propto \exp\{-(z + 2\eta\sigma)^2/2\}$$

So z has a Gaussian distribution with a mean $-2\eta\sigma$ and a standard deviation of one. Consequently,

$$25) \quad E\{z\pi(z\ a)\} = -2\eta\sigma$$

so from (4)

$$26) \quad d\,g(\sigma;\mu) = 2\eta\sigma d\sigma.$$

We note that this is independent of μ'. On integration and evaluation between 0 and σ_* we get

$$27) \quad h(\sigma_*) = \eta\sigma_*^2.$$

Dropping * then from (18) we observe that $R(\mu,\sigma) = \eta\sigma^2$. Equation (27) is confirmed by Fruend's results [5, p. 255].

Berhold obtained an estimate of $\eta = 4.25 \times 10^{-6}$ for a government contractor using the relationship in (27), [2, p. 101]. And this was used as a basis for determining the optimal sharing ratio in government incentive contracts [3].

If the probability distribution on x(a;z) has only one parameter then our problem is somewhat simplified. We let

$$28) \quad x[\mu, a;y] \equiv \mu - \mu(a) + y$$

where μ is an arbitrary constant and $\mu(a) = E(y|a)$. Since the probability distribution on y contains the parameter we make a transformation to remove it. For example if y has an exponential distribution with a mean of $\lambda > 0$ then we let $\lambda z = y$. Consequently

$$29) \quad x[\mu,\lambda,z] \equiv \mu - \lambda + \lambda z$$

where z has an exponential distribution with a mean of one. The decision maker can then choose λ.

Furthermore, if we use an exponential utility function:

$$30) \quad v(k) = 1 - \frac{1}{\phi} \exp[-\phi k], \; K > 0;$$

on the money amount k, then

$$31) \quad \pi(z|\lambda) \propto \exp\{-\phi[\mu - \lambda + \lambda z + h(\lambda)]\}\exp[-z]$$

$$= \exp\{-\phi\mu + \lambda\phi - \lambda\phi z - \phi h(\lambda) - z\}$$

$$\propto \exp\{-[\lambda\phi + 1]\, z\}$$

Using (4) and integrating from 0 to λ_*, we get

$$32) \quad h(\lambda_*) = \lambda_* - \{1/\phi\} \quad \{\mathrm{Ln}[\lambda_*\phi + 1]\}.$$

As before we will drop the * henceforth.

In the same manner as before we get

$$33) \quad c(a) = \mu + h(a) - R(a) = c(a^\circ) = \mu$$

where a° is the value of the parameter such that variance of y is zero. For the exponential distribution this would be where $\lambda = 0$. Thus we get $R(a) = h(a)$.

We observe $R(0) = 0$ and $R'(\lambda) > 0$ for all $\lambda > 0$. This result confirms our intuition. Intuitively we would also assume

$$34) \quad \frac{dR(\lambda)}{d\phi} < 0$$

Verbally the decision maker has a lower risk premium as ϕ increases. This would mean:

$$35) \quad \mathrm{Ln}[\lambda\,\phi + 1] > \frac{\lambda\,\phi}{\lambda\,\phi + 1}$$

On expanding the logarithmic series [6, p. 373] we get

$$36) \quad \mathrm{Ln}[\lambda\,\phi + 1] = \frac{\lambda\,\phi}{\lambda\,\phi + 1} + 1/2\left[\frac{\lambda\,\phi}{\lambda\,\phi + 1}\right]^2$$

$$+ 1/3\left[\frac{\lambda\,\phi}{\lambda\,\phi + 1}\right]^3 + \ldots$$

which confirms (35).

The results we have obtained thus far can be used as a basis for estimating utility functions along the lines suggested earlier. It would be useful to have other combinations of utility functions and probability distributions. This can be facilitated a great deal through the use of conjugate distributions [9, p. 47]. In addition, future research should also deal with more complicated payoff functions.

However, at this point it would seem that sufficient results have been presented so as to evaluate the usefulness of the basic concept. Even if the concept does not accurately reflect reality (in the sense that people do not actually have Gaussian utility functions) it should still be useful as a first approximation because of the computational simplicity.

FOOTNOTES

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